

PICARD-LEFSCHETZ THEOREM FOR FAMILIES OF NONSINGULAR ALGEBRAIC VARIETIES ACQUIRING ORDINARY SINGULARITIES

BY

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1. Introduction. The purpose of this paper is to expand the classical results of Picard [6] and Lefschetz [3, p. 23] concerning a one-complex-parameter family of algebraic varieties S_t acquiring singularities for isolated values of t . The question to be answered is: What action on $H_*(S_t)$ is induced by motion of the parameter t ? ($H_*()$ will denote (compact) singular homology with real coefficients.)

More precisely, let B be a complex analytic line bundle over a nonsingular complex projective variety X of (complex) dimension $(n+1)$. Let s_0 and s_∞ be any two linearly independent holomorphic sections of $B \rightarrow X$ such that $Z(s_\infty) \cap Z(ds_\infty) = \emptyset$. ($Z()$ will indicate "zeros of () in X ." Let $S_t = Z(s_0 + ts_\infty)$ and $S_\infty = Z(s_\infty)$. Then (s_∞/s_0) is a meromorphic and hence algebraic function on X , and so S_t is a projective hypersurface of X for all t . Since S_∞ is assumed nonsingular, standard arguments show that

$$C_0 = \{t \in \mathbb{C} : Z(s_0 + ts_\infty) \cap Z(ds_0 + t ds_\infty) \neq \emptyset\}$$

is finite.

For $t \in (\mathbb{C} - C_0)$, S_t has a C^∞ -normal bundle W in X whose fibres are ordinary two-real-dimensional discs. Let the fibration be given by $\omega: W \rightarrow S_t$. For t' near t , $S_{t'}$ is also nonsingular, and for each $P \in S_t$ the set $(\omega^{-1}(P) \cap S_{t'})$ contains exactly one point. Thus ω defines a diffeomorphism $F(t, t'): S_{t'} \rightarrow S_t$. Given any path γ in $(\mathbb{C} - C_0)$, beginning at t_0 and ending at t , we can define an associated diffeomorphism $F(\gamma) = F(t_k, t_{k-1}) \circ \cdots \circ F(t_2, t_1) \circ F(t_1, t_0)$ where $\{t_0, t_1, \dots, t_k = t\}$ is a sufficiently fine partition of γ . The homotopy type of $F(\gamma)$ does not depend on the choice of normal bundles [4, pp. 19–26], nor on the partition if it is sufficiently fine, but only on the homotopy class of γ in $(\mathbb{C} - C_0)$.

One asks, then, what is the associated homology isomorphism $F(\gamma)_*: H_*(S_{t_0}) \rightarrow H_*(S_t)$? From the discussion above, it clearly suffices to consider the case $0 \in C_0$, $\{t : 0 < |t| < 2\} \subseteq (\mathbb{C} - C_0)$, and $\gamma = \{\exp \xi : 0 \leq \xi \leq \theta\}$ where $\exp \xi = e^{2\pi i \xi}$. For such situations, define $F(\theta) = F(\gamma)$. In [3, p. 23], Lefschetz describes $F(q)_*: H_*(S_1) \rightarrow H_*(S_1)$ for $q \in \mathbb{Z}$, when S_0 has isolated double points as its singular locus. This result is variously called the Picard-Lefschetz Theorem or the Poincaré formula. Pham [5] extended these results to much more general types of isolated singularities. In the following we will generalize in another direction and allow $\{S_t\}$ to acquire an

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arbitrary ordinary singularity at S_0 . Our general philosophy will be to fibre the action of $F(\theta)$ over the singular locus of S_0 . (This approach also yields results in some simple cases of nonordinary, nonisolated singularities [1].) We will assume throughout that $\{0 < |t| < 2\} \subseteq (C - C_0)$.

Theorem 4.4 of this paper will give, from the topological point of view, an appropriate generalization of the classical Picard-Lefschetz Theorem to the case in which we acquire an arbitrary singularity (with arbitrary multiplicities) at S_0 . A. Landman, in his Berkeley thesis, treated this same problem from an algebraic point of view. To see the relation between these two approaches, we might note here a corollary of the homological dimensions of the subspaces of S_1 in Theorem 4.4 and Corollary 4.15:

For $0 \leq a \leq 2n$ and $m = \text{least common multiple of the multiplicities of the components of } S_0$ and

$$f_a = F(1)_* : H_a(S_1) \rightarrow H_a(S_1),$$

the minimal polynomial of the linear transformation f_a is a factor of the polynomial

$$(\lambda^m - 1)^{n+1-|a-n|}.$$

2. Ordinary singularities and the fibration of $F(\theta)$. For notational convenience, we will set $S_\infty = A(0)$. Then

DEFINITION 2.1. The family $\{S_t\}$ will be said to acquire an ordinary singularity at $t=0$ if:

- (1) $S_0 = A(1) \cup \dots \cup A(h)$ where the $A(i)$ are nonsingular hypersurfaces of X ;
- (2) for $0 \leq i < j \leq h$, $A(i)$ and $A(j)$ intersect normally (including the possibility that $A(i) \cap A(j) = \emptyset$).

From now on we assume that $\{S_t\}$ is as in §1 and acquires an ordinary singularity at $t=0$.

For any open set $U \subseteq X$, let $I(U) = \{i \in \{0, \dots, h\} : U \cap A(i) \neq \emptyset\}$. Then locally we can normalize our defining equation $s_0 + ts_\infty = 0$ for S_t :

LEMMA 2.2. *There is an open covering $\{U\}$ of $S_0 \cup S_\infty = A(0) \cup \dots \cup A(h)$ in X and complex analytic functions $x_i : U \rightarrow \mathbb{C}$ for each $U \in \{U\}$, $i \in I(U)$ such that*

- (1) $(x_i)_{i \in I(U)}$ are part of an analytic coordinate system for U in X ;
- (2) for each $U \in \{U\}$, $Z(x_i) = A(i) \cap U$;
- (3) $-((s_0/s_\infty) + t) = (\pi\{x_i^{m(i)} : i \in I(U)\} - t)$ on $U \in \{U\}$.

The $m(i)$ are integers whose values depend only on i and not on the open set U . $m(0) = -1$, and $m(i)$ is a positive integer for $i > 0$ and is called the multiplicity of $A(i)$ at $t=0$.

Proof. Theory of functions of several complex variables. See [2]. ■

DEFINITION 2.3. A finite open covering \mathcal{V} of $S_0 \cup S_\infty$ in X will be called a regular covering if

- (1) each $V \in \mathcal{V}$ is a simply connected subset of a coordinate disc in X ;
- (2) Lemma 2.2 holds for $\{U\} = \mathcal{V}$;

- (3) for $V \in \mathcal{V}, j \notin I(V), \bar{V} \cap A(j) = \emptyset$;
 (4) for $V, V' \in \mathcal{V}$ with $V \cap V' \neq \emptyset, V \cap V'$ is simply connected.
 Next define $A(I) = \bigcap \{A(i) : i \in I\}$ for $I \subseteq \{0, \dots, h\}$ and

$$B(q) = \bigcup \{A(I) : |I| = q\}.$$

THEOREM 2.4. *There exists a regular covering \mathcal{V} of $S_0 \cup S_\infty = B(1)$ and C^∞ -mappings:*

$$\begin{aligned} y(i, V) : (A(I(V)) \cap V) \times V &\rightarrow C \quad \text{for } i \in I(V), \\ z(V) : (A(I(V)) \cap V) \times V &\rightarrow C^{(n+1-|I(V)|)} \end{aligned}$$

for each $V \in \mathcal{V}$ such that:

- (1) $((y(i, V)(P, \quad)),_{i \in I(V)}, z(V)(P, \quad))$ forms a complex analytic coordinate system for V centered at P for each $P \in (A(I(V)) \cap V)$;
 (2) on $V \in \mathcal{V}, -(s_0/s_\infty + t) = (\pi\{y(i, V)(P, \quad))^{m(i)} : i \in I(V)\} - t)$;
 (3) for $V \cap V' \neq \emptyset$ and $I(V) = I(V')$:
 (a) $(y(i, V)(P_0, Q)/y(i, V')(P_0, Q))$ is a constant with absolute value 1 for each fixed $P_0 \in V \cap V' \cap A(I(V)), i \in I(V)$,
 (b) $Z(z(V)(P_0, Q)) = Z(z(V')(P_0, Q))$ for each P_0 ;
 (4) for $V \cap V' \neq \emptyset$ and $I(V) \subseteq I(V')$:
 (a) for $i \in I(V), (y(i, V)(Q_0, Q)/y(i, V')(P_0, Q))$ is a nonzero constant for $z(V)(Q_0, Q) = 0$ for all P_0, Q_0 fixed such that $Q_0 \in V \cap V' \cap A(I(V)), P_0 \in V' \cap A(I(V'))$, and $z(V')(P_0, Q_0) = 0$,

$$\begin{aligned} (b) \quad Z(z(V)(Q_0, Q)) &= Z(z(V')(P_0, Q)) \\ &\cap \bigcap \{Z(y(i, V')(P_0, Q) - y(i, V')(P_0, Q_0)) : i \in I(V') - I(V)\} \end{aligned}$$

for all P_0, Q_0 as in (a);

- (5) for all $V \in \mathcal{V}, V$ is the disjoint union of $Z(z(V)(P, Q))$ over $P \in (A(I(V)) \cap V)$.

Proof. Starting from $B(n+2) = \emptyset$, we shall assume that we have a finite cover \mathcal{V}_{q+1} of $B(q+1)$ which satisfies (2.3), (1)–(4), and (2.4), (1)–(5), and shall construct a cover \mathcal{V}_q with the corresponding properties over $B(q)$. Since $B(q) = \bigcup \{V : V \in \mathcal{V}_{q+1}\}$ is a disjoint union of subspaces, one for each I with $|I| = q$, it will clearly suffice to demonstrate the construction of the cover for $A(I) \cup B(q+1)$ for some I with $|I| = q$. Thus let \mathcal{V}' be a finite open covering of the compact set $A(I) - \bigcup \{V : V \in \mathcal{V}_{q+1}\}$ in X . By refining \mathcal{V}_{q+1} if necessary we can assume that

- (1) $I(V) = I$ for all $V \in \mathcal{V}'$ and $|I(V)| > q$ for all $V \in \mathcal{V}_{q+1}$,
 (2) $\mathcal{V}' \cup \mathcal{V}_{q+1}$ satisfies (2.3), (1)–(4) as a finite open covering of $A(I) \cup B(q+1)$ in X . (For (2.3)(4), use a covering of X by open stars with respect to a sufficiently fine triangulation.)
- Let $\mathcal{V}'' = \{V \in \mathcal{V}_{q+1} : I \subseteq I(V)\}$. For $V \in \mathcal{V}'$, let $(x(i, V))_{i \in I}$ be the coordinates as in Lemma 2.2. Let $w(V) : V \rightarrow C^{(n+1-q)}$ be such that $((x(i, V))_i, w(V))$ form a set of coordinates for V in X . Define $x(i, V)(P, Q) = x(i, V)(Q)$ for all $P \in A(I) \cap V$. Define $w(V)(P, Q) = w(V)(Q) - w(V)(P)$.

For $V \in \mathcal{V}''$, define

$$x(i, V)(P, Q) = y(i, V)(P_0, Q)$$

where $P \in V \cap A(I)$ and $z(V)(P_0, P) = 0$ and $i \in I$;

$$w(V)(P, Q) = (z(V)(P_0, Q), (y(j, V)(P_0, Q) - y(j, V)(P_0, P))_{j \in I(V) - I})$$

with P, P_0 as before.

Next for $V \in \mathcal{V}'$, $V' \in \mathcal{V}' \cup \mathcal{V}''$ and $V \cap V' \neq \emptyset$, define

$$\begin{aligned} u(i, V, V')(P, Q) \\ = (x(i, V')(P, Q) \cdot \pi\{y(j, V')(P_0, Q)^{m(j)/q(m(i))} : j \in I(V') - I\}) / x(i, V)(P, Q) \end{aligned}$$

for $i \in I$ with $P \in A(I) \cap V \cap V'$, $P_0 \in A(I(V')) \cap V'$ and $z(V')(P_0, P) = 0$. Further, for $V, V' \in \mathcal{V}' \cup \mathcal{V}''$, let $\Delta(V, V')(P)$ = the Jacobian matrix

$$(\partial w(V)(P, Q) / \partial w(V')(P, Q))|_P.$$

Now if $\{\rho_V\}$ is a C^∞ partition of unity of $A(I)$ subordinate⁽¹⁾ to the cover $\mathcal{V}' \cup \mathcal{V}''$ define:

$$y(i, V)(P, Q) = x(i, V)(P, Q) \cdot \pi\{u(i, V, V'')(P, Q)\rho_{V''}(P) : V'' \in \mathcal{V}' \cup \mathcal{V}''\}$$

for $V \in \mathcal{V}'$ and $i \in I$. Further define

$$\begin{aligned} z(V)(P, Q) \\ = w(V)(P, Q) - \sum \{\rho_{V''}(P)(w(V)(P, Q) - \Delta(V, V'')(P)w(V'')(P, Q)) : V'' \in \mathcal{V}' \cup \mathcal{V}''\}. \end{aligned}$$

With appropriate restrictions of various open sets of the coverings, these mappings give the induction step and hence the theorem. ■

The above theorem leads us to consider what happens locally with respect to normalized coordinates.

3. The normalized case. For the present chapter only, let $S_t = Z(y_1^{m_1} \cdots y_q^{m_q} - t) \subseteq \mathbb{C}^q$. Let m_c be the greatest common divisor of $\{m_1, \dots, m_q\}$. Then $S_t \approx T_t \times R$ for $t \neq 0$ where

$$\begin{aligned} T_t &= \{(y_1, \dots, y_q) \in S_t : |y_1|^{m_1} = \dots = |y_q|^{m_q}\}, \\ R &= \{(y_1, \dots, y_q) \in S_1 : y_i \text{ real, positive for all } i\}. \end{aligned}$$

LEMMA 3.1. For $t \neq 0$, T_t consists of m_c disjoint tori, each of real dimension $(q-1)$.

Proof. $(y_1^{m_1} \cdots y_q^{m_q}) - t$ factors into

$$\pi\{y_1^{n_1} \cdots y_q^{n_q} - t_k : 1 \leq k \leq m_c\}$$

where the t_k are the distinct m_c -roots of t and $n_i = (m_i/m_c)$. T_t has at least one component corresponding to each factor. But for the case $m_c = 1$, it is easily seen that T_t is pathwise connected. Hence in general T_t has exactly m_c components. An induction on q then gives that each factor is a $(q-1)$ -torus. ■

⁽¹⁾ See definition in [2, Appendix A].

It is immediate that $R \approx (R^+)^{q-1}$. Let

$$R_J = \{(r_1, \dots, r_q) \in R : r_i^{m_i} = r_j^{m_j} \text{ for all } i, j \in J, r_i^{m_i} \leq r_j^{m_j} \text{ for } i \in J, j \notin J\},$$

$$R_{J,\theta} = \{(y_1, \dots, y_q) : \exists (r_1, \dots, r_q) \in R_J, \theta_i \geq 0 \text{ for all } i \in J \text{ such that}$$

$$\sum \{m_i \theta_i : i \in J\} = \theta, y_i = (\exp \theta_i) r_i \text{ for } i \in J, y_i = r_i \text{ for } i \notin J\}.$$

Define $R_\theta = \bigcup \{R_{J,\theta} : J \subseteq \{1, \dots, q\}, J \neq \emptyset\}$. Then $R_\theta \subseteq S_t$ where $t = \exp \theta$, and R_θ is a continuous perturbation of $R = R_0$.

Further, for $J \subseteq \{1, \dots, q\}$, and a fixed constant $K > 1$, define

$$A(J) = \bigcap \{Z(y_i) : i \in J\} \subseteq C^\infty,$$

$$Y(J) = \{(y_1, \dots, y_q) : |y_i|^{m_i} < K, i \in J\},$$

$$Y'(J) = \{(y_1, \dots, y_q) \in Y(J) : |y_j|^{m_j} \geq K, j \notin J\},$$

$$A'(J) = A(J) \cap Y'(J),$$

and let $\mu(J) : \text{Cl}(Y'(J)) \rightarrow A'(J)$ be the natural projection produced by setting $y_i = 0$ for $i \in J$.

Define

$$\varphi : R \rightarrow Q = \{(\rho_1, \dots, \rho_q) \in R^q : \sum \rho_i = 1\}$$

by $\rho_i = (\log K - m_i \log r_i) / q \log K$. $\sigma_r = R \cap Y(\{1, \dots, q\})$ corresponds under φ to $\sigma_\rho = \{\rho \in Q : \rho_i \geq 0 \text{ for all } i\}$. For $J \subseteq \{1, \dots, q\}$, define $\sigma_\rho(J)$ to be the subsimplex spanned by the collection of vertices of σ_ρ which have 1 in the j th position for some $j \in J$. Let \bar{I} = the closed unit interval.

LEMMA 3.2. *There is a C^∞ retraction $\psi : Q \times \bar{I} \rightarrow Q$ of Q onto σ_ρ such that*

- (1) $\psi(\cdot, 0) = \text{identity on } Q$,
- (2) $\psi(\cdot, \alpha) = \text{identity on a neighborhood of } (1/q, \dots, 1/q) \text{ for all } \alpha \in \bar{I}$,
- (3) $\psi(\cdot, \alpha)(\sigma_\rho(J)) = \sigma_\rho(J) \text{ for all } J \subseteq \{1, \dots, q\}, \alpha \in \bar{I}$,
- (4) $\psi(\cdot, 1) : \varphi(Y'(J) \cap Q) \rightarrow \sigma_\rho(J) \text{ and this map is independent of the variables } r_j \text{ for } j \notin J$.

Proof. Use induction on q . (This retraction is similar to the one used by Pham [5, p. 338].) ■

Define $\rho : R \rightarrow \sigma_\rho$ to be the composition $\psi(\cdot, 1) \circ \varphi$. Then we can define $F(\theta) : S_1 \rightarrow S_{\exp \theta}$ by

$$F(\theta)(y) = (\exp(\rho_1 \theta / m_1) y_1, \dots, \exp(\rho_q \theta / m_q) y_q)$$

where $(\rho_1, \dots, \rho_q) = \rho(|y_1|, \dots, |y_q|)$. $F(\theta)$ is induced by a C^∞ -normal bundle construction as in §1 and its action is fibred over $A'(J)$ by the fibrations $\mu(J)$ defined above. Further $F(\theta)(R)$ is homotopic to R_θ in $S_{\exp \theta}$ in such a way that

- (1) the homotopy respects the fibration $\mu(J)$ of $S_{\exp \theta} \cap Y'(J)$;
- (2) under the homotopy, $F(\theta)(\sigma_r)$ corresponds to

$$\sigma_\theta = R_\theta \cap \text{Cl}(Y(\{1, \dots, q\})).$$

If we orient $\sigma_r = \sigma_0$ by the natural order of local coordinates r_1, \dots, r_{q-1} , then $F(\theta)$ induces an orientation on each

$$\sigma_{J,\theta} = R_{J,\theta} \cap \text{Cl}(Y(\{1, \dots, q\})) \text{ of } \sigma_\theta.$$

For $J = \{1, \dots, q\}$, this orientation is equal to $(-1)^{q-1}$ times the orientation induced by the natural order of the local coordinates $\theta_1, \dots, \theta_{q-1}$. This follows from Lemma 3.2(2) and the fact that under $F(\theta)$ we have $(\partial\theta_i/\partial r_i) < 0$ at $(1/q, \dots, 1/q)$.

Let m be any common multiple of $\{m_1, \dots, m_q\}$. For $J = \{1, \dots, q\}$, we consider σ_θ as a $(q-1)$ -homology chain with orientation and coefficients of its simplicial decomposition given by considering σ_θ as the image under $F(\theta)$ of the oriented simplex σ_0 . Then $\sigma_{J, m(k-1)}$ is parametrized by

$$\{(\theta_1, \dots, \theta_{q-1}) : \theta_i \geq 0 \text{ for all } i, \sum \{m_i \theta_i : 1 \leq i \leq q-1\} \leq mk \text{ and } m_{i_0} \cdot \theta_{i_0} \geq m\}$$

for any fixed $1 \leq i_0 \leq q-1$. Using this fact and direct calculation one obtains that the chain

$$\eta_{m, q-1} = \sigma_{m(q-1)} - \binom{q-1}{1} \sigma_{m(q-2)} + \binom{q-1}{2} \sigma_{m(q-3)} - \dots \pm \binom{q-1}{q-1} \sigma_{m \cdot 0}$$

is a $(q-1)$ -chain parametrized by $0 \leq m_i \theta_i \leq m$, $1 \leq i \leq q-1$.

Let T be the $(q-1)$ -torus in T_1 which has nonempty intersection with R . T is a topological group with a component T' of $\{(y_1, \dots, y_q) \in T : y_{q-1} = 1\}$ as a maximal connected subgroup, and

$$\{(y_1, \dots, y_q) \in T : y_1 = \dots = y_{q-2} = 1 \text{ and } y_{q-1} = \exp \theta_{q-1}, \\ 0 \leq \theta_{q-1} \leq (\text{g.c.d. } \{m_1, \dots, m_{q-2}, m_q\} / m_c)\}$$

a transverse set for T/T' . Thus by our remarks on orientation above and induction on q we have

FORMULAE 3.3. $\eta_{m, q-1} = (-1)^{q-1} (m^{q-1} m_c / \pi \{m_i : 1 \leq i \leq q\}) \cdot T$ and if S_1 is oriented by its complex structure, then as an intersection number formula:

$$\sigma_0 \cdot (\eta_{m, q-1}) = (-1)^{q(q-1)/2} (m^{q-1} \cdot m_c / \pi m_i).$$

Finally, define

$$T_i(J) = \{(y_1, \dots, y_q) \in S_i : |y_i|^{m_i} = |y_j|^{m_j} \text{ for } i, j \in J\}.$$

From the considerations of Lemma 3.1 it is clear that there exists an $m(J)$ -sheeted complex analytic covering space $C_i(J)$ of $A'(J)$, where $m(J) = \text{g.c.d. } \{m_i : i \in J\}$, and the following commutative diagram of fibrations:

$$(3.4) \quad \begin{array}{ccc} S_i \cap \text{Cl}(Y'(J)) & \subseteq & \text{Cl}(Y'(J)) \\ \downarrow \tau(J) & & \downarrow \mu(J) \\ T_i(J) & & \\ \downarrow \pi(J) & \searrow \lambda(J) & \\ C_i(J) & \longrightarrow & A'(J) \end{array}$$

where π has fibre a $(q'-1)$ -real torus and τ has fibre a $(q'-1)$ -simplex, $q' = |J|$. Then for $|J| = q-1$, $t=1$, and $j_0 \notin J$, $\lambda(J)^{-1}(P \in A'(J) : |y_{j_0}|^{m(j_0)} = K)$ has m_c components and for $C =$ any one of these components:

(3.5) $\pi(J)^{-1}(C) \approx$ (one of the $(q-1)$ -tori of T_1), with this isomorphism given by the projection $\tau(I) : S_1 \cap \text{Cl}(Y'(\{1, \dots, q\})) \rightarrow T_1$ where $|I| = q$.

4. Vanishing bundles and the action of $F(\theta)$. We now return to the general problem considered in §1 and §2.

THEOREM 4.1. *There are tubular neighborhoods $Y(i)$ of $A(i)$ in X for $0 \leq i \leq h$ such that for $Y(I) = \bigcap \{Y(i) : i \in I\}$ there is a C^∞ -normal bundle fibration*

$$\mu(I) : \text{Cl}(Y(I)) \rightarrow A(I).$$

Further the following conditions are satisfied:

(1) For $W(q) = \bigcup \{Y(I) : |I| = q\}$ and $Y'(I) = Y(I) - W(|I| + 1)$, $A'(I) = A(I) - W(|I| + 1)$, one has that $\mu(I)^{-1}(A'(I)) = \text{Cl}(Y'(I))$.

(2) For each $P \in A'(I)$, there is a $V \in \mathcal{V}$ with $I(V) = I$ and $\mu(I)^{-1}(P) \subseteq V$, where \mathcal{V} is the regular covering of Theorem 2.4.

(3) For $J \subseteq I$, $\mu(I) \circ \mu(J) = \mu(I)$ on $\text{Cl}(Y(I))$.

(4) For each $I \subseteq \{0, \dots, h\}$ with $|I| = q$, there is a constant $K(I) > 0$ such that, for $P \in A'(I)$ and $V \ni \mu^{-1}(P)$, the mapping $\mu^{-1}(P) \rightarrow \mathbb{C}^q$ given by $(y(i, V)(P, Q))_{i \in I}$ is an isomorphism of $\mu^{-1}(P)$ onto the closed polycylinder with radii $K^{1/m(i)}$ in \mathbb{C}^q .

(5) $\mu(I)^{-1}(P) \cap A'(J)$ is a $(|I| - |J|)$ -real torus for $P \in A'(I)$, $J \subseteq I$.

(6) $(W(q) - \text{Cl}(W(q+1)))$ is a disjoint union of open sets, one for each I with $|I| = q$.

Proof. To construct $\mu(I)$ use Theorem 2.4, parts 3(b), 4(b), and (5). Then all the properties of the theorem follow from Theorem 2.4 and the behavior of normalized coordinates as seen in §3. ■

Next let us construct $F(\theta) : S_1 \rightarrow S_{\exp \theta}$ which is induced by a normal bundle construction as in §1 and which behaves like the $F(\theta)$ constructed in §3 on each $\mu(I)^{-1}(P)$ for $P \in A'(I)$. Let $\delta_i : X \rightarrow [0, 1]$ be a C^∞ -function such that:

(1) $\delta_i \equiv 1$ in a neighborhood of $A(i)$;

(2) $\delta_i \equiv 0$ on $X - Y(i)$;

(3) for $P \in A'(I)$, $i \in I$, and $Q, Q' \in \mu^{-1}(P) \subseteq V$ with

$$|y(i, V)(P, Q)| = |y(i, V)(P, Q')|$$

we have $\delta_i(Q) = \delta_i(Q')$. For t sufficiently small, we have for each $Q \in S_t$ that $\delta_i(Q) = 1$ for some $i \in \{1, \dots, h\}$. We can assume that we have this condition for $|t| < 2$. Define, for $Q \in S_1$,

$$\rho_i(Q) = (\delta_i(Q) / \sum \{\delta_j(Q) : 1 \leq j \leq h\}) - \delta_0(Q)$$

for $1 \leq i \leq h$ and $\rho_0(Q) = \delta_0(Q)$.

Define $F(\theta, P): \mu(I)^{-1}(P) \cap S_1 \rightarrow \mu(I)^{-1}(P) \cap S_{\exp \theta}$ for $P \in A'(I)$ by:

$$F(\theta, P)((y(i, V)(P, Q))_{i \in I}) = ((\exp(\rho_i(Q)\theta/m(i))y(i, V)(P, Q))_{i \in I})$$

where $V \supseteq \mu(I)^{-1}(P)$. Theorem 2.4, parts (3) and (4), shows that this definition is independent of the choice of V and that the $F(\theta, P)$ piece together to give a C^∞ -mapping:

$$(4.2) \quad F(\theta): S_1 \rightarrow S_{\exp \theta}.$$

This mapping is easily seen to be induced by a normal bundle construction as described in §1. Hence we can assume that, whenever $\{S_t\}$ acquires an ordinary singularity at $t=0$, then $F(\theta)$ is as constructed in (4.2).

We now proceed to introduce the other constructions used in §3 to the general case with the help of normalized coordinates and the fibrations $\mu(I)$.

On $\mu(I)^{-1}(P)$ with $P \in A'(I)$, we will let $(y_i)_{i \in I}$ be any of the coordinate systems $(y(i, V)(P, \cdot))_{i \in I}$ for $V \supseteq \mu^{-1}(P)$ and $I(V)=I$. By Theorem 2.4, the choice of V will not matter in what follows.

For $0 \notin I$, define

$$T_t(I) = \{Q \in S_t \cap Y'(I) : |y_i|^{m(i)} = |y_j|^{m(j)} \text{ for } i, j \in I\}.$$

Then for t sufficiently small, hence we assume for $0 < |t| < 2$, $T_t(I) \cap \mu(I)^{-1}(P)$ is, by Lemma 3.1, $m(I)$ real tori each of real dimension $(|I|-1)$, for $P \in A'(I)$. ($m(I)$ = greatest common divisor of $\{m(i) : i \in I\}$.) Then there exists a complex manifold $C_t(I)$, which is an $m(I)$ -sheeted covering space of $A'(I)$ and such that $T_t(I)$ is a $(q-1)$ -torus bundle over $C_t(I)$, where $q=|I|$.

Further we can define a fibration $\tau(I): (S_t \cap \text{Cl}(Y'(I))) \rightarrow T_t(I)$ by

$$\tau(I)^{-1}(Q) = \{Q' \in (S_t \cap \mu(I)^{-1}(P)) : \text{Arg } y_i(Q') = \text{Arg } y_i(Q) \text{ for all } i \in I\}$$

where $P = \mu(I)(Q)$. Then we have a commutative diagram of projections for $0 < |t| < 2$:

$$(4.3) \quad \begin{array}{ccc} S_t \cap \text{Cl}(Y'(I)) & \subseteq & \text{Cl}(Y'(I)) \\ \downarrow \tau(I) & & \downarrow \mu(I) \\ T_t(I) & & \\ \downarrow \pi(I) & \nearrow \lambda(I) & \\ C_t(I) & \longrightarrow & A'(I), \end{array}$$

which is a replica of (3.4) on each $\mu(I)^{-1}(P)$, $P \in A'(I)$.

By (3.5) and Theorem 4.1(5), $\pi(I): T_1(I) \rightarrow C_1(I)$ can be factored into a series of circle bundles. Hence the composition of the Gysin maps associated to each of the circle bundles gives maps:

$$\begin{array}{ccccc} H_*(C_1(I)) & \longrightarrow & H_{*+(q-1)}(T_1(I)) & \longrightarrow & H_{*+(q-1)}(S_1), \\ \downarrow & & \downarrow & & \downarrow \\ H_*(C_1(I), \partial C_1(I)) & \rightarrow & H_{*+(q-1)}(T_1(I), \text{Cl}(W(q+1))) & \rightarrow & H_{*+(q-1)}(S_1, \text{Cl}(W(q+1))), \end{array}$$

where, intuitively, the horizontal maps take a cycle of $C_1(I)$ into the fibre above it in $T_1(I)$. Since it will cause no confusion, we will denote both of the horizontal composition maps by $\psi(I)$.

By §3, $\tau(I)$ has fibre a $(q-1)$ -simplex so that we can apply the Thom-Gysin isomorphism and construct:

$$H_*(S_1) \rightarrow H_*(S_1, K(I)) \approx H_{*-(q-1)}(T_1(I), \text{Cl}(W(q+1)))$$

where $K(I) = \text{Cl}(S_1 - Y'(I))$. Call this composition $\chi(I)$.

Now let m = a common multiple of $\{m(1), \dots, m(h)\}$. Define $\eta_m = F(m)_* - (\text{identity}) : H_*(S_1) \rightarrow H_*(S_1)$ where $F(\theta)_*$ is as in §1 and hence can be assumed to be defined by $F(\theta)$ in (4.2). Since there is little danger of confusion, we shall use η_m also to refer to the analogous

$$F(m)_* - (\text{identity}) : H_*(S_1) \rightarrow H_*(S_1, \text{Cl}(W(q+1))).$$

We are now ready to state and prove a generalized Poincaré formula.

Let $\Gamma_p(I)$ denote a basis for $H_p(C_1(I), \partial C_1(I))$, and let $\Gamma_{p'}(I)$ denote its dual basis in $H_{p'}(C_1(I))$ where $p+p' = 2(n-q+1)$. Assume $C_1(I)$ to be oriented from the complex structure of $A'(I)$. For $\gamma \in \Gamma_p(I)$ we shall denote its dual by γ' . Finally let $\mathcal{J}_q = \{I \subseteq \{1, \dots, h\} : |I| = q\}$.

THEOREM 4.4. *The mapping $(\eta_m)^{q-1} : H_{p+q-1}(S_1) \rightarrow H_{p+q-1}(S_1, \text{Cl}(W(q+1)))$ is given by*

$$(\eta_m)^{q-1}(\tilde{\alpha}) = (-1)^r m^{q-1} \sum \{(m(I)/\pi_I m(i))(\tilde{\alpha} \cdot \psi(\gamma'))\psi(\gamma) : \gamma \in \Gamma_p(I), I \in \mathcal{J}_q\}$$

where $r = (q(q-1)/2) + p(q-1)$.

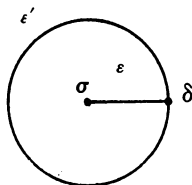
Proof. For $0 \in I$, let $T_1(I) = C_1(I) = A'(I)$ and define $\tau(I) = \mu(I)|_{S_1 \cap \text{Cl}(Y'(I))}$ $\pi(I) = \lambda(I) = \text{identity map}$. We will need a series of lemmas.

LEMMA 4.5. *For $0 \in I$, $P \in A(I)$, $\mu(I)^{-1}(P) \cap S_1$ is an analytic polydisc of complex dimension $|I| - 1$.*

Proof of lemma. By Definition 2.1, $A(I)$ is a nonsingular subvariety of S_1 . Further $\mu(I) : Y(I) \rightarrow A(I)$ is a normal bundle in X . Hence each $\mu(I)^{-1}(P)$ must intersect S_1 normally. ■

LEMMA 4.6. *There exists a finite cellular decomposition of $B(1) = S_0 \cup S_\infty$ such that:*

(1) *The decomposition respects the chain of subspaces $B(1) \supseteq B(2) \supseteq \dots \supseteq B(n+2) = \emptyset$ as well as each of the subspaces $A'(I)$, $I \subseteq \{0, \dots, h\}$;*



(2) for a p -cell σ of $A'(I)$, there is a $V \in \mathcal{V}$ (see Theorem 2.4) with $\mu(I)^{-1}(\sigma) \subseteq V$ and $I = I(V)$; and for $\varepsilon = \{Q \in (A(J) \cap \text{Cl}(Y'(I)) : y_j(Q) \text{ real } \geq 0\}$ and

$$\varepsilon' = \{Q \in (A(J) \cap \text{Cl}(Y'(I)) : |y_j(Q)| = K(I)^{1/m(j)}\}$$

and $\delta = \varepsilon \cap \varepsilon'$ with $J \cup \{j\} = I$, we have $\varepsilon, \varepsilon' \subseteq (p+1)$ -skeleton of $A(J)$, $\delta \subseteq p$ -skeleton of $A'(J)$;

(3) the decomposition lifts through λ to a cellular decomposition of each $C_1(I)$.

Proof of lemma. (1) and (3) are immediate from the elementary theory of C - W complexes. For (2) use Theorem 4.1(5). ■

Define

$$Y_k = \bigcup \{\pi(I)^{-1}(\pi(I)^{-1}(p\text{-skeleton of } C_1(I))) : I \subseteq \{0, \dots, h\}, p + |I| = k\}.$$

LEMMA 4.7. The natural mapping $H_{k-1}(Y_k) \rightarrow H_{k-1}(S_1)$ is onto.

Proof of lemma. Let $K_g = (S_1 - W(g)) \cup Y_k$ and assume $H_{k-1}(K_{g+1}) \rightarrow H_{k-1}(S_1)$ is onto. We have an exact sequence:

$$H_{k-1}(K_g) \rightarrow H_{k-1}(K_{g+1}) \rightarrow H_{k-1}(K_{g+1}, K_g).$$

However by Lemma 4.6(2), and using the Thom-Gysin isomorphism for $\tau(I)$ (see (4.3) and Lemma 4.5), we have for $p = k - g$ and $T_1(I)_p = \pi(I)^{-1}(p\text{-skeleton of } C_1(I))$:

$$H_{k-1}(K_{g+1}, K_g) \approx \sum \{H_p(T_1(I), T_1(I)_p) : I \subseteq \{0, \dots, h\}, |I| = g\}.$$

Using that $\tau(I)$ can be factored into circle bundles, each with an ordinary Gysin sequence, it is easily concluded that:

$$H_p(T_1(I), T_1(I)_p) \approx H_p(C_1(I), p\text{-skeleton}) = 0.$$

Hence $H_{k-1}(K_g)$ is onto by the composition of two surjective maps. ■

LEMMA 4.8. The natural mapping $H_{k-1}(Y_k) \rightarrow H_{k-1}(Y_k, Y_{k-1})$ is injective.

Proof of lemma. It suffices to show that $H_{k-1}(Y_{k-1}) = 0$. Define

$$L_g = \bigcup \{\tau(I)^{-1}(\pi(I)^{-1}(p\text{-skeleton of } C_1(I))) : p + |I| = k - 1 \text{ and } |I| \geq g\}.$$

Suppose $H_{k-1}(L_{g+1}) = 0$. L_g has as a strong deformation retract the space

$$L_{g+1} \cup \bigcup \{T_1(I)_p : |I| = g\}$$

for $p = k - 1 - g$. However each $T_1(I)$ has dimension at most $p + (|I| - 1) = k - 2$. Hence $H_{k-1}(L_g, L_{g+1}) = 0$ and therefore $H_{k-1}(L_g) = 0$. ■

$F(m)$ as constructed in (4.2) respects all the subspaces constructed above. Hence by Lemmas 4.7 and 4.8, it suffices to characterize

$$\eta_m = F(m)_* - (\text{identity}) : H_{k-1}(Y_k, Y_{k-1}) \rightarrow H_{k-1}(Y_k, Y_{k-1}).$$

Let σ be a p -cell of $C_1(I)$ for $p + |I| = k$. Let $M = \tau(I)^{-1}(\pi(I)^{-1}(\sigma))$. Then the boundary of M , which we shall denote by M' , is equal to the union of

$N = \tau(I)^{-1}(\pi(I)^{-1}(\sigma'))$ and $\tilde{G} = M - W(|I|)$. Finally let $G = \tilde{G} \cap Y_{k-1}$. Then $H_{k-1}(Y_k, Y_{k-1})$ is the direct sum of groups of the form:

$$(4.9) \quad H_{k-1}(M, G \cup N);$$

hence it suffices to consider $\eta_m: H_{k-1}(M, G \cup N) \rightarrow H_{k-1}(M, G \cup N)$, since each subspace of this type is preserved by $F(m)$ as constructed in (4.2). Further we have the exact sequence:

$$(4.10) \quad H_{k-1}(\tilde{G}, G) \rightarrow H_{k-1}(M, G \cup N) \rightarrow H_{k-1}(M, \tilde{G} \cup N).$$

Using Lemma 4.6(2), the generator for $H_{k-1}(M, \tilde{G} \cup N)$ can be represented by the cell

$$(4.11) \quad \omega = \{Q \in M : y_i(Q) = r_i \bar{y}_i \text{ for all } i \in I \text{ for some } (r_i) \in R\}$$

where R is as in §3 and $\bar{y}_i = y_i(\bar{Q})$ for any fixed $\bar{Q} \in \pi(I)^{-1}(\sigma)$. For the proper choice of \bar{Q} , $\omega' \in G \cup N$ by the construction of the cellular decomposition in Lemma 4.6(2).

(4.12) Since $H_{k-1}(\tilde{G}, G)$ is itself a direct sum of groups of the form (4.9), but over p -cells σ' of $C_1(I')$ with $p' + |I| = k$ and $|I'| < |I|$ we can inductively assume that any element of $H_{k-1}(\tilde{G}, G)$ has a representative ω' which is a sum of cells of type (4.11) with real coefficients over p' -cells σ' with $p' > p$.

By (4.10), for any cycle α of $H_{k-1}(M, G \cup N)$, there is a representative chain of the form $a(\sigma)\omega + \omega'$ with ω and ω' as in (4.11) and (4.12) and $a(\sigma) \in \mathbf{R}$. Now by (4.2) and the first of Formulae 3.3, $(F(m) - (\text{identity}))^{q-2}(\omega')$ is homologous to a sum of chains of the form $\pi(I')^{-1}(\sigma')$ where σ' is as in (4.12) and $q = |I|$. But $F(m)$ as constructed in (4.2) is homotopic in each fibre to the identity on each $\pi(I')^{-1}(\sigma')$. Hence $(F(m) - (\text{identity}))^{q-1}(\omega')$ is homologous to 0 in $H_{k-1}(M, G \cup N)$. Further by the same formula we have that up to sign:

$$(4.13) \quad (F(m) - (\text{identity}))^{q-1}(\omega) \sim (m^{q-1}m(I)/\pi_I m(i))T_\sigma \text{ if } I \in \mathcal{J}_q, \text{ where}$$

$$T_\sigma = \{Q \in M : |y_i(Q)|^{m(i)} = |y_j(Q)|^{m(j)} \text{ for all } i, j \in I\},$$

and $(F(m) - (\text{identity}))^{q-1}(\omega) \sim 0$ otherwise. (\sim indicates homologous to in $H_{k-1}(M, G \cup N)$.)

Now let $\tilde{\alpha} \in H_{p+q-1}(S_1)$. Then $\tilde{\alpha}$ has a representative cycle α in Y_{p+q} which is made up of cells of the form $a(\sigma)\omega$ with ω as in (4.11). Further, for $I \in \mathcal{J}_q$,

$$\sum \{a(\sigma)\sigma : \sigma \text{ } p\text{-cell of } C_1(I)\}$$

is, with appropriate orientation, the cycle of $H_p(C_1(I), \text{Cl}(W(q+1)))$ given by $\pi(I)(\chi(I)(\tilde{\alpha}))$. ($\chi(I)$ was defined earlier in §4.) The theorem then follows from (4.13) and the fact that orientation for the mapping $\chi(I)$ can be chosen so that:

$$((\tilde{\alpha} \cdot (\psi(I)(\gamma')))) \text{ in } S_1 = (-1)^{p(q-1)}((\pi(I)(\chi(I)(\tilde{\alpha}))) \cdot \gamma')$$

in $C_1(I)$). The calculation of the sign in the theorem is then essentially given by the second formula of Formulae 3.3. This completes the proof of Theorem 4.4. ■

Note 4.14. Let $H_0(I)$ be the bundle associated to $\lambda \circ \pi: T_1(I) \rightarrow A'(I)$ with fibre $H_0((\lambda_0 \pi)^{-1}(P))$ for $P \in A'(I)$, (see [7, p. 151]). Then

$$\begin{aligned} H_p(C_1(I)) &\approx H_p(A'(I); H_0(I)), \\ H_p(C_1(I), \partial C_1(I)) &\approx H_p(A'(I), \partial A'(I); H_0(I)). \end{aligned}$$

■

COROLLARY 4.15. *Let α be an element of $H_{k-1}(S_1 - W(q+1))$. Then*

$$\eta_m^{q-1}(\alpha) = (-1)^r m^{q-1} \sum \{(m(I)/\pi_I m(i))(\alpha \cdot \psi(\gamma))\psi(\gamma') : \gamma' \in \Gamma'_p(I), I \in \mathcal{J}_q\}$$

in $H_{k-1}(S_1)$ where $r = (q(q-1)/2) + pq$.

Proof. Choose a representative cycle for α in $(Y_k - W(q+1))$ and proceed as in Theorem 4.4. Note that $(\gamma' \cdot \gamma) = (-1)^p$. ■

In the case $q=1$, Theorem 4.4 is true if we define η_m^0 to be the standard homology map

$$H_*(S_1) \rightarrow H_*(S_1, \text{Cl}(W(2))).$$

However this provides no new information. For the sake of completeness, we shall add here the more appropriate result for the case $q=1$. For $I \in \mathcal{J}_1$ and

$$M(I) = \{\exp(k/m(I)) : k \in \mathbb{Z}\}$$

with generator $\xi(I) = \exp(1/m(I))$, we have that $M(I)$ acts on $S_1 \cap \text{Cl}(Y'(I))$ as in (4.2). For $\gamma \in H_*(C_1(I), \partial C_1(I))$, $\psi(I)(\gamma) = \gamma$. Let

$$\tilde{\psi}(I)(\gamma) = (\xi(I) \cdot \gamma) - \gamma.$$

As in [5], the group-ring $\mathbb{Z}(M(I))$ acts on $H_*(C_1(I), \partial C_1(I))$. Define $\tilde{k} = \exp(k/m(I))$.

THEOREM 4.16. *For $\alpha \in H_p(S_1)$ and $b \geq 1$, then, in $H_p(S_1, \text{Cl}(W(2)))$,*

$$\eta_b(\alpha) = \sum \{(\alpha \cdot \gamma') \tilde{k} \cdot \tilde{\psi}(\gamma) : 0 \leq k \leq b-1, \gamma \in \Gamma_p(I), I \in \mathcal{J}_1\}.$$

In particular, if b is a multiple of $m(i)$ for all $1 \leq i \leq h$, then $\eta_b(\alpha) = 0$ in $H_p(S_1, \text{Cl}(W(2)))$.

Proof. Immediate from the fact that $\eta_b = F(b)_* - (\text{identity})$ and from the construction of $F(\theta)$ in (4.2). Note that, for

$$\xi = \sum \{\tilde{k} : 1 \leq k \leq m(I)\} \in \mathbb{Z}(M(I)),$$

we have $\xi \cdot \psi(\gamma) = 0$ for all $\gamma \in \Gamma_p(I)$. ■

COROLLARY 4.17 *Let α be an element of $H_p(S_1 - \text{Cl}(W(2)))$. Then*

$$\eta_b(\alpha) = (-1)^p \sum \{(\alpha \cdot \gamma') \tilde{k} \cdot \tilde{\psi}(\gamma') : 0 \leq k \leq b-1, \gamma' \in \Gamma'_p(I), I \in \mathcal{J}_1\}$$

in $H_p(S_1)$. ■

5. Applications. The proof of Theorem 4.4 actually shows the action of $F(l)_* : H_*(S_1) \rightarrow H_*(S_1)$ for any $l \geq 0$. For if $\alpha \in H_{k-1}(S_1)$, then there is a $\beta \in H_{k-1}(Y_k, Y_{k-1})$ which has a common preimage with α in $H_{k-1}(Y_k)$. Further, β has as representative a sum of cells of the form $a_\omega \cdot \omega$ with $a_\omega \in \mathbf{R}$ and ω as in (4.11). Then $F(l)_*(\beta)$ is represented in $H_{k-1}(Y_k, Y_{k-1})$ by the sum of cells $a_\omega \cdot \omega_i$ where

$$\omega_i = \{Q \in M : y_i(Q) = r_i \bar{y}_i \text{ for all } i \in I \text{ for some } (r_j) \in R_i\}$$

with M as in (4.9) and R_i as in §3. By Lemma 4.8, such an element of $H_{k-1}(Y_k, Y_{k-1})$ comes from a unique element of $H_{k-1}(Y_k)$ which in turn determines $F(l)_*(\alpha) \in H_{k-1}(S_1)$. However, in general this is an unwieldy process.

In this final section we will give a formula which helps characterize $F(m)_* : H_*(S_1) \rightarrow H_*(S_1)$ without necessitating the explicit construction outlined above. (m here, as before, is a common multiple of $m(i)$ for $1 \leq i \leq h$.) With this formula and Theorem 4.4 we will completely construct

$$\eta_m = F(m)_* - (\text{identity}) : H_*(S_1) \rightarrow H_*(S_1)$$

for the case $n=2$. (The case $n=0$ is trivial and for $n=1$, which corresponds to the classical Picard-Lefschetz formula, η_m is completely given by Theorem 4.4.)

FORMULA 5.1. Let α be an element of $H_{p+q-1}(S_1 - W(q+2))$. Then for all $\beta \in H_{p+q-1}(S_1)$ with $p+p'=2(n-q)$:

$$\begin{aligned} & (\eta_m^{q-1}(\alpha) \cdot \beta) + (-1)^q(\alpha \cdot \eta_m^{q-1}(\beta)) \\ &= (-1)^r(q-1)m^q \sum \{(m(I)/\pi_I m(i))(\alpha \cdot \psi(\gamma))(\psi(\gamma') \cdot \beta) : \gamma \in \Gamma_p(I), I \in \mathcal{J}_{q+1}\} \end{aligned}$$

where $r = ((q+1)q/2) + p(q+1)$.

Proof. By Corollary 4.15, the right-hand side of the formula is simply $(q-1)\eta_m^q(\alpha) \cdot \beta$. However $\eta_m^q(\alpha)$ is invariant under $F(m)_*$ and $F(m)$ is an orientation-conserving automorphism. Hence $\eta_m^q(\alpha) \cdot \beta = \eta_m^q(\alpha) \cdot F(m)_*^k(\beta)$ for all $k \in \mathbf{Z}$. Thus the right-hand side of the formula is equal to

$$\sum \{\eta_m^q(\alpha) \cdot F(m)_*^k(\beta) : 1 \leq k \leq q-1\}$$

which in turn is equal to:

$$\sum \left\{ (-1)^{(q-j)} \binom{q}{j} F(m)_*^j(\alpha) \cdot F(m)_*^k(\beta) : 0 \leq j \leq q, 1 \leq k \leq q-1 \right\}.$$

This last summation reduces to $\eta_m^{q-1}(\alpha) \cdot \beta + (-1)^q(\alpha \cdot \eta_m^{q-1}(\beta))$ by repeated use of the equality:

$$F(m)_*^j(\alpha) \cdot F(m)_*^k(\beta) = F(m)_*^{j+i}(\alpha) \cdot F(m)_*^{k+i}(\beta) \quad \text{for all } i \in \mathbf{Z}. \quad \blacksquare$$

Now let us examine $\eta_m : H_j(S_1) \rightarrow H_j(S_1)$ for $n=2$. For $j=0$ and 4, $\eta_m \equiv 0$ trivially. For $j=1$, η_m is given by Corollary 4.15. For $j=3$, η_m is given by Theorem 4.4 since the natural map $H_3(S_1) \rightarrow H_3(S_1, \text{Cl}(W(3)))$ is an injection. We shall now

characterize $\eta_m: H_2(S_1) \rightarrow H_2(S_1)$. Note that $W(4) = \emptyset$ and so for any $\alpha, \beta \in H_2(S_1)$ we have that:

$$\eta_m(\alpha) \cdot \beta + \alpha \cdot \eta_m(\beta) = (-1)m^2 \sum \{(m(I)/\pi_1 m(i))(\alpha \cdot \psi(\gamma))(\beta \cdot \psi(\gamma)) : \gamma \in \Gamma_0(I), I \in \mathcal{J}_3\}$$

by Formula 5.1. Define the right-hand side of this expression to be equal to $\Phi(\alpha, \beta)$. This formula has as corollaries that:

(5.2) For all $\alpha, \beta \in H_2(S_1)$: $\eta_m(\alpha) \cdot \eta_m(\beta) = -\Phi(\alpha, \beta)$ since $\alpha \cdot \eta_m^2(\beta) = \Phi(\alpha, \beta)$ by Theorem 4.4;

(5.3) $\eta_m^2(\alpha) = 0$ for $\alpha \in H_2(S_1)$ if and only if $\eta_m(\alpha) \cdot \eta_m(\alpha) = 0$ (since $\eta_m(\alpha) \cdot \eta_m(\alpha) = 0$ implies $(\alpha \cdot \psi(\gamma)) = 0$ for $\gamma \in \Gamma_0(I)$, $I \in \mathcal{J}_3$).

Let

$$E = \bigcup \{A(I) : I \in \mathcal{J}_3\}.$$

(E is called the set of triple points of the hypersurface S_0 .) Let $B = \lambda^{-1}(E)$ and for each $P \in C(I)$, $I \in \mathcal{J}_3$, let $T(P) = \pi(I)^{-1}(P)$, $W(P) = \tau(I)^{-1}(\pi(I)^{-1}(P))$. Let \bar{Q}_P be fixed in $T(P)$ and let ω_P be as in (4.11). Let

$$S'_1 = (S_1 - W(3)) \cup \bigcup \{\omega_P : P \in B\}.$$

Then by the Thom-Gysin isomorphism and Lemma 4.5, $H_2(S'_1) \rightarrow H_2(S_1)$ is onto. Let

$$D(J) = \pi(J)(\bigcup \{T_1(J) \cap \omega_P : P \in B\})$$

and let

$$K = \bigcup \{\tau(J)^{-1}(\pi(J)^{-1}(Q)) : Q \in D(J)\}.$$

Then, as in §4, we define for $J \in \mathcal{J}_2$:

$$\begin{aligned} \psi(J) &: H_1(C_1(J), D(J)) \rightarrow H_2(S_1 - W(3), K) \\ \tilde{\chi}(J) &: H_2(S'_1) \rightarrow H_1(C_1(J), D(J)) \end{aligned}$$

with orientations chosen as in the final paragraph of the proof of Theorem 4.4. ($\tilde{\chi}(J)$ corresponds to $\pi(J) \circ \chi(J)$ of §4.)

For each $P \in B$, let τ_P be a generator of $H_2(W(P)) = H_2(T(P))$. Let $\{\tau_1, \dots, \tau_s\}$ be a maximal linearly independent set in $\{\tau_P : P \in B\}$ and let $\{\alpha_1, \dots, \alpha_s\}$ be a dual set under the intersection pairing in $H_2(S_1)$.

LEMMA 5.4. *There are unique elements $\eta_i \in H_2(S_1)$ for $1 \leq i \leq s$ such that there exist elements $\xi_P \in H_2(W(P), K)$ for $P \in B$ such that:*

(1) if $\varphi_i = -m \sum \{(m(J)/\pi_1 m(j))\psi(J) \circ \tilde{\chi}(J)(\alpha'_i) : J \in \mathcal{J}_2\}$ where α'_i is some pre-image of α_i in $H_2(S'_1)$, then the image of η_i in $H_2(S_1, K)$ is given by

$$\varphi_i + \sum \{(\alpha_i \tau_P) \xi_P : P \in B\};$$

(2) for $1 \leq i, j \leq s$, $\eta_i \alpha_j + \alpha_i \eta_j = \Phi(\alpha_i, \alpha_j)$.

Proof. There is a representative of α'_i of the form

$$\varepsilon + \sum \{(\alpha_i \cdot \tau_P) a_P \omega_P : P \in B\}$$

where ε is a cycle of $(S_1 - W(3))$ relative to K and the a_P are constants in R . Then $F(m)$ fixes $(S_1 - W(3))$, K , and $W(P)$ for each P by (4.2), and so:

(1) $\eta_m: H_2(S_1 - W(3), K) \rightarrow H_2(S_1 - W(3), K)$ takes $\{\varepsilon\}$ into φ_i by the proof of Theorem 4.4;

(2) $\eta_m: H_2(W(P), K) \rightarrow H_2(W(P), K)$ takes $\{a_P \omega_P\}$ into an element of $H_2(W(P), K)$ which we will call ξ_P .

Then putting $\eta_i = \eta_m(\alpha_i)$, the existence of a collection $\{\eta_i\}$ satisfying the lemma is proved. For uniqueness, suppose η_i, η'_i for $1 \leq i \leq s$ satisfy the conditions of the lemma. Let ξ_P, ξ'_P be the corresponding elements of $H_2(W(P), K)$ for all $P \in B$. Then

$$\eta_i - \eta'_i = \sum \{(\alpha_i \cdot \tau_P)(\xi_P - \xi'_P) : P \in B\}$$

in $H_2(S_1, K)$. However $H_2(K) = 0$ and $\partial_*(\xi_P - \xi'_P) = 0$ in $H_1(K)$ and so $(\xi_P - \xi'_P)$ comes from an element $b_P \tau_P$ in $H_2(S_1)$. To get the uniqueness of the η_i it suffices to show that for $1 \leq i \leq s$:

$$(*) \quad \sum \{(\alpha_i \cdot \tau_P) b_P \tau_P : P \in B\} = 0.$$

But by condition (2) of the lemma:

$$(\eta_i - \eta'_i) \cdot \alpha_j + \alpha_i \cdot (\eta_j - \eta'_j) = 0.$$

Hence for all $1 \leq j \leq s$, we have that:

$$2 \sum \{b_P (\alpha_i \cdot \tau_P) (\alpha_j \cdot \tau_P) : P \in B\} = 0.$$

Since the α_j form a dual basis to the subspace of $H_2(S_1)$ generated by $\{\tau_P : P \in B\}$, this suffices to prove (*). ■

COROLLARY 5.5. $\eta_i = \eta_m(\alpha_i) \in H_2(S_1)$ for all $1 \leq i \leq s$. ■

Now $\Gamma_1(J)$, the basis for $H_1(C_1(J), \partial C_1(J))$ for $J \in \mathcal{J}_2$, can be so chosen that:

$$\Gamma_1(J) = \Delta_1(J) \cup \Omega_1(J), \quad \Gamma'_1(J) = \Delta'_1(J) \cup \Omega'_1(J)$$

where $\Delta_1(J)$ generates the image of $H_1(C_1(J))$ in $H_1(C_1(J), \partial C_1(J))$ and $\Omega'_1(J)$ generates the image of the mapping $H_1(\partial C_1(J)) \rightarrow H_1(C_1(J))$ and $\gamma \in \Delta_1(J)$ implies that $\gamma' \in \Delta'_1(J)$. For any $\alpha \in H_2(S_1)$,

$$\alpha'' = \alpha - \sum \{(\alpha \cdot \tau_i) \alpha_i : 1 \leq i \leq s\}$$

comes from an element (which we shall also call α'') of $H_2(S_1 - W(3))$. By Corollary 4.15,

$$\eta_m(\alpha'') = (-1)m \sum \{(m(J)/\pi_J m(j))(\alpha'' \cdot \psi(\gamma))\psi(\gamma') : \gamma' \in \Gamma'_1(J), J \in \mathcal{J}_2\}$$

in $H_2(S_1)$. Now $\Gamma'_1(J) = \Delta'_1(J) \cup \Omega'_1(J)$ and for $\gamma' \in \Omega'_1(J)$, $\psi(\gamma')$ is contained in the subspace of $H_2(S_1)$ generated by $\{\tau_P : P \in B\}$ by (3.5). Hence

$$\begin{aligned} \eta_m(\alpha'') &= (-1)m \sum \{(m(J)/\pi_J m(j))(\alpha'' \cdot \psi(\gamma))\psi(\gamma') : \gamma \in \Delta_1(J), J \in \mathcal{J}_2\} \\ &\quad + \sum \{b_i(\alpha'') \tau_i : 1 \leq i \leq s\}. \end{aligned}$$

However by Formula 5.1 and Corollary 5.5, $(\alpha'' \cdot \eta_i) + (\eta_m(\alpha'') \cdot \alpha_i) = \Phi(\alpha'', \alpha_i) = 0$ for all $1 \leq i \leq s$. Thus intersecting this last formula for $\eta_m(\alpha'')$ with α_i we have that:

$$b_i(\alpha'') = m \sum \{(m(J)/\pi_J m(j))(\alpha'' \cdot \psi(\gamma))(\alpha_i \cdot \psi(\gamma')) : \gamma \in \Delta_1(J), J \in \mathcal{J}_2\} - (\alpha'' \cdot \eta_i).$$

Define

$$\tilde{\psi}(\gamma) = \psi(\gamma) - \sum \{(\alpha_i \cdot \psi(\gamma))\tau_i : 1 \leq i \leq s\} \in H_2(S_1)$$

for $\gamma \in \Delta_1(J)$, $J \in \mathcal{J}_2$. Then $\alpha \cdot \tilde{\psi}(\gamma) = \alpha'' \cdot \psi(\gamma)$. Define

$$\begin{aligned} \tilde{\eta}_i &= \eta_i - m \sum \{(m(J)/\pi_J m(j))(\alpha_i \cdot \psi(\gamma'))\tilde{\psi}(\gamma) : \gamma \in \Delta_1(J), J \in \mathcal{J}_2\} \\ &\quad - \sum \{(\alpha_k \cdot \eta_i)\tau_k : 1 \leq k \leq s\} \end{aligned}$$

THEOREM 5.6 For $\alpha \in H_2(S_1)$, then

$$\begin{aligned} \eta_m(\alpha) &= \sum \{(\alpha \cdot \tau_i)\eta_i : 1 \leq i \leq s\} \\ &\quad - m \sum \{(m(J)/\pi_J m(j))(\alpha \cdot \tilde{\psi}(\gamma))\psi(\gamma') : \gamma \in \Delta_1(J), J \in \mathcal{J}_2\} \\ &\quad - \sum \{(\alpha \cdot \tilde{\eta}_i)\tau_i : 1 \leq i \leq s\} \end{aligned}$$

in $H_2(S_1)$, where m is a common multiple of $m(j)$ for all $1 \leq j \leq h$.

Proof. The formula is an immediate consequence of the formulae and definitions just preceding. ■

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⁽²⁾ (In the above reference, the author's treatment of the triple point case is incorrect.)